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Micromechanics-based thermoviscoelastic constitutive equations for rubber-like matrix composites at finite strains

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Abstract

Global constitutive equations that model the response of multiphase materials undergoing finite deformations in which any phases behaves as a rubber-like thermoviscoelastic material are derived. The monolithic thermoviscoelastic constituent is modeled by a free-energy function which is given by a sum of a long-term contribution, that is based on the entropic elasticity for thermoelastic polymers, plus a non-equilibrium part which characterizes the viscoelastic (dissipative) mechanism. The global constitutive relations that govern the behavior of the composite are derived by using a micromechanical analysis in conjunction with the homogenization technique. Applications are given that illustrate the response of a rubber-like thermoviscoelastic matrix reinforced by continuous elastic nylon fibers. Results exhibit the effect of the viscoelasticity on the response of the composite when it is subjected to thermal and mechanical loadings, as well as its creep and relaxation behavior at room and elevated temperature.

Keywords: Finite deformation; Entropic elasticity; Thermoviscoelasticity; Micromechanics; Homogenization

1. Introduction

Rubber-like materials are described by the so-called entropic elasticity according to which the deformation is associated with a significant change of entropy but with very small change in internal energy. This is in contrast to the behavior of metals, glass and ceramics which are described by energetic elasticity according to which the deformation is associated with significant changes of internal energy. A detailed discussion of entropic and energetic elasticity is given by Holzapfel (2000). Rubbers exhibit a distinct behavior referred to as the Gough–Joule effect. In ordinary materials (e.g., metals and ceramics) that are subjected to prescribed extensions, the gradient of stress with respect to temperature is always negative. In rubber-like material on the other hand, this gradient becomes positive at prescribed extensions beyond a critical level, thereafter increasing with extension. Similarly, in ordinary materials that are subjected to

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prescribed loadings, the gradient of deformation with respect to temperature is always positive. In rubber-like materials this gradient becomes negative for loadings beyond a critical value, thereafter decreasing with loading. Thus, beyond this critical value, called the thermoelastic inversion point, the rubber has a negative coefficient of thermal expansion. This anomalous behavior of rubber-like solids can be demonstrated by hanging a weight on a strip of rubber and changing the temperature. The weight will rise upwards when the rubber is heated and it will lower when it is cooled. Consequently, free-energy functions which are required to model the thermoelastic behavior of rubber-like materials must provide this effect. Such free-energy functions have been proposed by Chadwick (1974), Chadwick and Creasy (1984), Ogden (1992), Morman (1995), Holzapfel (2000) and Horgan and Saccomandi (2003) for example.

In the rubbery state at low temperatures, rubbers do not exhibit viscoelastic effects such as creep and relaxation. At elevated temperature on the other hand a temperature-dependent viscoelastic response is obtained as it was observed, for example, in vulcanized rubber in the temperature range of 100–150 °C (Tobolsky et al., 1944). Hence free-energy functions that are capable to describe the thermoviscoelastic behavior at finite strains of rubber-like materials at elevated temperatures must be determined. Such energy functions have been recently proposed by Holzapfel and Simo (1996), Lion (1997) and Reese and Govindjee (1998). The free-energy function of Holzapfel and Simo (1996) is given as a sum of the long-term behavior of the material and a non-equilibrium part that depends on deformation, temperature and a set of internal variables. The specific form of the free-energy that characterizes the non-equilibrium state has been determined from the restrictions imposed by the second law of thermodynamics which implies that the dissipation is non-negative. The stress tensor that is derived from the free-energy function is given a sum of the stress tensor at equilibrium plus a non-equilibrium part. The latter is governed by a set of evolution equations in time. A rheological interpretation can be given to this thermoviscoelastic free-energy function by considering the generalized one-dimensional Maxwell model. In this case the internal variables correspond to the viscous strains while the non-equilibrium stresses correspond to the viscous stresses.

The previous discussion concerns with the modeling of monolithic thermoviscoelastic rubber-like materials. In order to establish constitutive equations for multiphase composites undergoing large deformations in which one or more phases are thermoviscoelastic rubber-like material, a finite strain micromechanical analysis must be followed. For finite thermoelastic rubber-like matrix composites (namely in the absence of viscoelastic effects) a micromechanical procedure, based on the homogenization technique for periodic microstructures, has been carried out by Aboudi (2002) by the analysis of a periodic unit cell from which the corresponding global constitutive equations were derived. This was achieved by establishing the mechanical and thermal concentration tensors that relate the local deformation gradient increment to the global externally applied one. Once these concentration tensors have been established, the derivation of the overall constitutive relations of the multiphase material can be readily determined. For papers in which the homogenization procedure is employed to examine the large mechanical deformations of composite materials, see Agah-Tehrani (1990) and Takano et al. (2000), for example.

In a recent publication by Aboudi and Pindera (2004), the capability of the aforementioned micromechanical model, referred to as high-fidelity generalized method of cells, to predict the response of continuously reinforced elastic composites undergoing finite deformations, has been demonstrated. This has been achieved by comparisons with exact elasticity solutions (Horgan, 1995) for a porous composite with four different types of matrix material under axisymmetric loading, and a finite-element analysis of a repeating unit cell representative of a unidirectionally-reinforced periodic composite subjected to transverse loading. The comparisons of the predicted overall responses and internal fields with the exact and finite element solutions exhibit excellent agreements.

In the present investigation, the thermoviscoelastic constitutive equations at finite strains of multiphase materials in which any one of phases is governed by the thermoviscoelastic free-energy function of Holzapfel and Simo (1996) for rubber-like materials, are micromechanically established. To this end, the analysis of Aboudi (2002) is generalized in order to establish the mechanical, thermal and viscoelastic tensors that relate the increments of the local deformation gradient and the externally applied global deformation gradient. These provide, in conjunction with the definition of average quantities, the requested incremental constitutive equations of the composite. The micromechanical analysis is based on the homogenization technique of periodic multiphase materials, and is carried out by identifying a repeating unit cell that is divided into several generic cells. Each generic cell is further divided into four subcells in which a quadratic expansion of the displacement increment is assumed. An averaging procedure establishes a set of incremental relations which are based on the satisfaction of the local equilibrium equations, interfacial traction and displacement continuity conditions, and the satisfaction of periodic boundary conditions, see Aboudi (2002) for more details.

The resulting global constitutive relations are implemented to study the behavior of a thermoviscoelastic rubber-like matrix reinforced by continuous elastic nylon fibers. The long-term behavior is characterized by a thermoelastic matrix which is modeled by a free-energy function that is based on the developments of Chadwick (1974), Chadwick and Creasy (1984) and Morman (1995). Results are given that illustrate the effect of the viscoelastic mechanism on the response caused by thermal loading in the presence of constant applied stretches and constant applied stresses both of which are applied in the transverse (perpendicular) direction to the long fibers. In addition, the behavior of the composite under free thermal loading is shown. The response of the composite to mechanical loadings, its creep under constant stress loading and its relaxation under constant stretching are exhibited at room and elevated temperature.

2. The finite monolithic thermoviscoelastic material

2.1. Helmholtz free-energy function

In this subsection the constitutive relations of the monolithic (unreinforced) thermoviscoelastic material undergoing finite deformations that have been developed by Holzapfel and Simo (1996) are briefly presented.

Let $\mathbf{F}$ denote the deformation gradient from which the right Cauchy-Green deformation tensor $\mathbf{C} = \mathbf{F}^t\mathbf{F}$, where $\mathbf{F}^t$ denotes the transpose of $\mathbf{F}$, is determined. The Helmholtz free-energy per unit volume is given by

$$\psi(\mathbf{C}, \theta, \mathbf{\Gamma}_\alpha) = \psi^\infty(\mathbf{C}, \theta) + \sum_{\alpha=1}^{N_\alpha} \Upsilon^\alpha(\mathbf{C}, \theta, \mathbf{\Gamma}_\alpha) \tag{1}$$

where ψ^∞ is the long-term free-energy at equilibrium (at time $t \to \infty$) when the material responds perfectly elastic, $\sum_{\alpha=1}^{N_\alpha} \Upsilon^\alpha$ represents the free-energy that characterizes the non-equilibrium state which provides the creep and relaxation behavior and θ is the temperature. The second-order tensors $\mathbf{\Gamma}_\alpha, \alpha = 1, \ldots, N_\alpha$, are N_α internal variables (viscous strains-like) that describe the irreversible process. These tensors have the same role as the deformation tensor $\mathbf{C}$ (from which the strain tensor is derived).

The long-term second Piola–Kirchhoff stress tensor $\mathbf{S}^\infty$ at equilibrium and the non-equilibrium stresses $\mathbf{Q}_\alpha$ (viscous stresses-like) are given by

$$\mathbf{S}^\infty = 2\frac{\partial \psi^\infty}{\partial \mathbf{C}} \tag{2}$$

$$\mathbf{Q}_\alpha = 2\frac{\partial \Upsilon^\alpha}{\partial \mathbf{C}} = -\frac{\partial \Upsilon^\alpha}{\partial \mathbf{\Gamma}_\alpha} \tag{3}$$

Consequently, the total second Piola–Kirchhoff stress tensor $\mathbf{S}$ is determined from

$$\mathbf{S} = \mathbf{S}^{\infty} + \sum_{\alpha=1}^{N_\alpha} \mathbf{Q}_\alpha \tag{4}$$

Let $\psi^\alpha(\mathbf{C}, \theta)$ denote the free-energy associated with the viscoelastic contribution to the α-process. The corresponding second Piola–Kirchhoff stress tensor $\mathbf{S}_\alpha$ is obtained as follows

$$\mathbf{S}_\alpha = 2\frac{\partial \psi^\alpha}{\partial \mathbf{C}} \tag{5}$$

From the second law of thermodynamics, which requires that the internal dissipation is non-negative, Holzapfel and Simo (1996) obtained the following expression for the dissipative functions Υ^α:

$$\Upsilon^\alpha = \mu_\alpha |\mathbf{\Gamma}_\alpha|^2 - 2\frac{\partial \psi^\alpha}{\partial \mathbf{C}} \mathbf{\Gamma}_\alpha + \psi^\alpha \tag{6}$$

where μ_α defines a non-negative temperature-dependent parameter. This is followed by the evolution equations for $\mathbf{Q}_\alpha$ (which are motivated by the linear generalized Maxwell model) that are given by

$$\dot{\mathbf{Q}}_\alpha + \frac{\mathbf{Q}_\alpha}{\tau_\alpha} = \dot{\mathbf{S}}_\alpha - 2\dot{\mu}_\alpha \mathbf{\Gamma}_\alpha, \quad \mathbf{Q}_\alpha|_{t=0} = 0 \tag{7}$$

where a dot denotes a derivative with respect to time t,

$$\dot{\mu}_\alpha = \frac{\partial \mu_\alpha}{\partial \theta}\dot{\theta}$$

and τ_α are relaxation times. The latter depend on the temperature and are commonly characterized by the Arrhenius exponential equation. The initial conditions in Eq. (7) indicate that at time $t = 0$ the viscoelastic stresses vanish. The second term on the right-hand-side of Eq. (7), $2\dot{\mu}_\alpha \mathbf{\Gamma}_\alpha$, contains temperature-dependent material parameters and vanishes for isothermal processes.

By employing the first relation in Eq. (3) the following expression for $\mathbf{Q}_\alpha$ is obtained:

$$\mathbf{Q}_\alpha = 2\frac{\partial \psi^\alpha}{\partial \mathbf{C}} - 4\frac{\partial^2 \psi^\alpha}{\partial \mathbf{C}\partial \mathbf{C}} \mathbf{\Gamma}_\alpha \tag{8}$$

On the other hand, by employing the second relation in Eq. (3), the following expression can be established:

$$\mathbf{Q}_\alpha = -2\mu_\alpha \mathbf{\Gamma}_\alpha + 2\frac{\partial \psi^\alpha}{\partial \mathbf{C}} \tag{9}$$

It follows from Eqs. (8) and (9) that the parameter $2\mu_\alpha$ can be chosen as:

$$2\mu_\alpha \mathbf{I}^{(4)} = 4\frac{\partial^2 \psi^\alpha}{\partial \mathbf{C}\partial \mathbf{C}} \tag{10}$$

where $\mathbf{I}^{(4)}$ is the fourth-order identity tensor ($I^{(4)}_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2$, with δ_{ij} denoting the Kronecker delta).

For viscoelastic polymers that are composed of identical chains it can be assumed (Holzapfel and Simo, 1996) that

$$\psi^\alpha = \omega_\alpha^\infty \psi^\infty \tag{11}$$

where ω_α^∞ are given non-dimensional parameters. It follows from Eq. (10) that

$$2\mu_\alpha \mathbf{I}^{(4)} = \omega_\alpha^\infty \mathbf{D}^\infty \tag{12}$$

where $\mathbf{D}^{\infty}$ is the instantaneous fourth-order tangent tensor of the material at equilibrium ($t \to \infty$):

$$\mathbf{D}^{\infty} = 4\frac{\partial^2 \psi^{\infty}}{\partial \mathbf{C} \partial \mathbf{C}} \tag{13}$$

In conclusion, the thermoviscoelastic material at finite strain is characterized by the long-term free-energy function ψ^{∞} in conjunction with the dissipative functions given by Eqs. (6), (11) and (12), and the evolution equation (7). To this end, one needs to specify in addition to the functional form of ψ^{∞} the temperature-dependent parameters τ_{α} and the non-dimensional parameters ω_{α}^{∞}.

2.2. Long-term free-energy for the finite deformations of rubber-like solids

The structure of free-energy functions of rubber-like solids are based on entropic elasticity (Holzapfel, 2000) and must provide the so called Gough–Joule effect. In the present investigation, the free-energy function for the large deformation a rubber-like solid that has been established by Chadwick (1974), Chadwick and Creasy (1984) and Morman (1995) is employed to model the long-term free-energy function ψ^{∞} of the finite thermoviscoelastic material. It has also been employed by Aboudi (2002) to analyze the behavior of thermoelastic rubber-like matrix composites. It is given by

$$\psi^{\infty}(\mathbf{C},\theta) = \kappa\left[g(J)\frac{\theta}{\theta_0} - \alpha_0 h(J)(\theta-\theta_0)\right] + G\left[f(\mathbf{C})\frac{\theta}{\theta_0} - \gamma l(\mathbf{C})\left(\frac{\theta}{\theta_0}-1\right)\right] + \psi_2(\theta) - \psi_2(\theta_0)\frac{\theta}{\theta_0} \tag{14}$$

where κ, G, α_0 are the initial bulk, shear and volume coefficient of thermal expansion, respectively, $\gamma(0 \leqslant \gamma 1)$ is a non-dimensional scalar and ψ_2 is a function of temperature with θ_0 being a reference temperature. The volumetric response functions $g(J)$ and $h(J)$, where $J = \det \mathbf{F} > 0$, can be determined from the pressure–volume–temperature relation of the thermoelastic material. These functions are given by Chadwick (1974) in the form

$$g(J) = \frac{1}{m}\left[J + \frac{J^{1-m}}{m-1} - \frac{m}{m-1}\right] \quad (m > 1) \tag{15}$$

$$h(J) = \frac{1}{n}(J^n - 1) \quad (n > 1) \tag{16}$$

The distortional functions $f(\mathbf{C})$ and $l(\mathbf{C})$ were chosen by Morman (1995) as follows

$$\begin{aligned} f(\mathbf{C}) = \frac{1}{G}\Big[&c_{10}(I_1 - 3J^{2/3}) + c_{01}(I_2 - 3J^{4/3}) + c_{11}(I_1 - 3J^{2/3})(I_2 - 3J^{4/3}) + c_{20}(I_1 - 3J^{2/3})^2 \\ &+ c_{30}(I_1 - 3J^{2/3})^3\Big] \end{aligned} \tag{17}$$

where c_{10}, c_{01}, c_{11}, c_{20}, c_{30} are material constants, and I_1 and I_2 are the first and second invariants of $\mathbf{C}$, namely,

$$I_1 = \text{tr}\,\mathbf{C} = C_{ii}$$

$$I_2 = \frac{1}{2}(\text{tr}^2\mathbf{C} - \text{tr}\,\mathbf{C}^2)$$

and

$$l(\mathbf{C}) = (1-\gamma)f(\mathbf{C})$$

It should be noted that under dilatational loading in which $F_{11} = F_{22} = F_{33}$ it can be readily verified that $f(\mathbf{C})$ vanishes, as well as the derivatives of $f(\mathbf{C})$ with respect to $\mathbf{C}$. It is worth mentioning that both the bulk and shear moduli depend, according to Eq. (14), linearly on the temperature θ. This is consistent with the Gaussian statistical theory of molecular networks, see Holzapfel (2000).

The second Piola–Kirchhoff stress tensor $\mathbf{S}^{\infty}$ and the fourth-order tangent tensor $\mathbf{D}^{\infty}$ are obtained from ψ^{∞} by employing Eqs. (2) and (13), respectively. The free-energy functions ψ^{α} are determined from ψ^{∞} by employing Eq. (11).

Finally, the long-term thermal stress tensor γ^{∞} is obtained from ψ^{∞} or $\mathbf{S}^{\infty}$ by employing the following relations

$$\gamma^{\infty} = -2\frac{\partial^2 \psi^{\infty}}{\partial \mathbf{C} \partial \theta} = -\frac{\partial \mathbf{S}^{\infty}}{\partial \theta} \tag{18}$$

With the derived expressions for $\mathbf{D}^{\infty}$ and γ^{∞} from the free-energy function ψ^{∞} as given by Eqs. (13) and (18), respectively, the following incremental constitutive equation is readily obtained from Eq. (2):

$$\Delta \mathbf{S}^{\infty} = \frac{1}{2}\mathbf{D}^{\infty}\Delta\mathbf{C} - \gamma^{\infty}\Delta\theta \tag{19}$$

where $\Delta\theta$ is the deviation of the temperature θ from the reference temperature θ_0. This equation expresses the increment of the second Piola–Kirchhoff stress tensor at equilibrium in terms of the deformation and temperature increments.

2.3. Recurrence formula for the evolution of the non-equilibrium stresses

Elimination of the internal strains $\mathbf{\Gamma}_{\alpha}$ from Eq. (9) and their substitution in Eq. (7) in conjunction with Eq. (5), provides the following form of the evolution equations for the non-equilibrium stresses:

$$\dot{\mathbf{Q}}_{\alpha} + \left(\frac{1}{\tau_{\alpha}} - \frac{\dot{\mu}_{\alpha}}{\mu_{\alpha}}\right)\mathbf{Q}_{\alpha} = \dot{\mathbf{S}}_{\alpha} - \frac{\dot{\mu}_{\alpha}}{\mu_{\alpha}}\mathbf{S}_{\alpha} \tag{20}$$

so that

$$\mathbf{Q}_{\alpha}(t) = \int_0^t \frac{\mu_{\alpha}(t)}{\mu_{\alpha}(s)} \exp\left[-\frac{t-s}{\tau_{\alpha}}\right]\left[\dot{\mathbf{S}}_{\alpha}(s) - \frac{\dot{\mu}_{\alpha}(s)}{\mu_{\alpha}(s)}\mathbf{S}_{\alpha}(s)\right]\mathrm{d}s \tag{21}$$

The value of $\mathbf{Q}_{\alpha}(t+\Delta t)$, where Δt is a time increment, can be obtained from Eq. (21) by expressing the integral in this equation as a sum of integrals from $t = 0$ to t and from t to $t + \Delta t$. As a result, the following approximate recurrence formula can be established:

$$\mathbf{Q}_{\alpha}(t+\Delta t) \approx \exp\left[-\frac{\Delta t}{\tau_{\alpha}}\right]\mathbf{Q}_{\alpha}(t) + \exp\left[-\frac{\Delta t}{2\tau_{\alpha}}\right]\left[\mathbf{S}_{\alpha}(t+\Delta t)\left(1 - \frac{\Delta t}{2}\frac{\dot{\mu}_{\alpha}(t+\Delta t)}{\mu_{\alpha}(t+\Delta t)}\right) - \mathbf{S}_{\alpha}(t)\left(1 + \frac{\Delta t}{2}\frac{\dot{\mu}_{\alpha}(t)}{\mu_{\alpha}(t)}\right)\right] \tag{22}$$

with

$$\mathbf{S}_{\alpha}(t) = \omega_{\alpha}^{\infty}\mathbf{S}^{\infty}(t) \tag{23}$$

This recurrence formula eliminates the need to store tensor quantities at all previous time steps. It is referred to by Simo and Hughes (1998) as a one-step unconditionally stable recurrence formula.

2.4. Incremental constitutive relations

The use of the aforementioned constitutive equations that model the finite deformation of a thermo-viscoelastic material in the micromechanical analysis that will be described in the sequel leads to a large system of non-linear algebraic equations. It is more efficient to represent these equations in an incremental form. This will be shown to yield a system of linear algebraic equations which can be easily handled.

The increment of the second Piola–Kirchhoff stress tensor is given according to Eq. (4) by

$$\Delta\mathbf{S}(t) = \mathbf{S}^{\infty}(t) - \mathbf{S}^{\infty}(t-\Delta t) + \sum_{\alpha=1}^{N_\alpha}[\mathbf{Q}_\alpha(t) - \mathbf{Q}_\alpha(t-\Delta t)] = \Delta\mathbf{S}^{\infty}(t) + \sum_{\alpha=1}^{N_\alpha}\Delta\mathbf{Q}_\alpha(t) \tag{24}$$

By employing Eqs. (22) and (23) the following expression for the increment $\Delta\mathbf{S}(t)$ is obtained

$$\Delta\mathbf{S}(t) = \xi\Delta\mathbf{S}^{\infty}(t) - \frac{\Delta t}{2}[\eta(t)\mathbf{S}^{\infty}(t) + \eta(t-\Delta t)\mathbf{S}^{\infty}(t-\Delta t)] - \sum_{\alpha=1}^{N_\alpha}\zeta_\alpha\mathbf{Q}_\alpha(t-\Delta t) \tag{25}$$

where

$$\eta(t) = \sum_{\alpha=1}^{N_\alpha}\omega_\alpha^{\infty}\exp\left[-\frac{\Delta t}{2\tau_\alpha}\right]\frac{\dot{\mu}_\alpha(t)}{\mu_\alpha(t)}, \quad \xi = 1 + \sum_{\alpha=1}^{N_\alpha}\omega_\alpha^{\infty}\exp\left[-\frac{\Delta t}{2\tau_\alpha}\right], \quad \zeta_\alpha = 1 - \exp\left[-\frac{\Delta t}{\tau_\alpha}\right]$$

Since the micromechanical analysis is based on the actual stress, let us employ the following relation that provides the first (non-symmetric) Piola–Kirchhoff stress tensor **T** in terms of the second Piola–Kirchhoff stress tensor **S**

$$\mathbf{T} = \mathbf{S}\mathbf{F}^{t} \tag{26}$$

By using the expression of $\Delta\mathbf{S}$ given by Eq. (25) and the relation for $\Delta\mathbf{S}^{\infty}$ given by (19), one can, after some manipulations, establish the following incremental constitutive law which governs the response of the thermoviscoelastic material at finite strains:

$$\Delta\mathbf{T} = \mathbf{R}\;\Delta\mathbf{F} - \mathbf{H}\;\Delta\theta - \Delta\mathbf{V} \tag{27}$$

where **R** is the current fourth-order tangent tensor which is given by

$$R_{ijkl} = \xi D_{iplq}^{\infty}(t)F_{jp}(t)F_{kq}(t) + \delta_{jk}\left[S_{il}^{\infty}(t) + \sum_{\alpha=1}^{N_\alpha}Q_{(\alpha)il}(t)\right] \tag{28}$$

H is the second-order thermal stress tensor:

$$H_{ij} = \xi\gamma_{ip}^{\infty}(t)F_{jp}(t) \tag{29}$$

and $\Delta\mathbf{V}$ is the second-order viscoelastic tensor that accounts for the history of deformation:

$$\Delta V_{ij} = \sum_{\alpha=1}^{N_\alpha}\zeta_\alpha Q_{(\alpha)ip}(t-\Delta t)F_{jp}(t) + \frac{\Delta t}{2}\left[\eta(t)S_{ip}^{\infty}(t) + \eta(t-\Delta t)S_{ip}^{\infty}(t-\Delta t)\right]F_{jp}(t) \tag{30}$$

It should be noted that since Δt is a small time increment, the order of magnitude of ζ_α is $O(\Delta t)$, which together with the second term on the right-hand-side of Eq. (30) justifies representing its left-hand-side as an incremental quantity.

3. Homogenization

Consider a multiphase composite with a periodic microstructure in the $X_2 - X_3$ plane as shown in Fig. 1 where the repeating unit cell used to construct the periodic array is highlighted. In the framework of the homogenization method, the displacement increment $\Delta\mathbf{u}$ is asymptotically expanded in terms of a small parameter δ as follows:

$$\Delta\mathbf{u}(\mathbf{X}) = \Delta\mathbf{u}_0(\mathbf{X},\mathbf{Y}) + \delta\Delta\mathbf{u}_1(\mathbf{X},\mathbf{Y}) + \cdots \tag{31}$$

where $\mathbf{X} = (X_1, X_2, X_3)$ are the initial macroscopic (global) coordinate system, and $\mathbf{Y} = (Y_1, Y_2, Y_3)$ are the microscopic (local) initial coordinates that are defined with respect to the repeating unit cell. The size of the unit cell is further assumed to be much smaller than the size of the body so that the relation between the global and local systems is

$$\mathbf{Y} = \frac{\mathbf{X}}{\delta} \tag{32}$$

where δ is a small scaling parameter characterizing the size of the unit cell. This implies that a movement of order unity on the local scale corresponds to a very small movement on the global scale (for more details about the homogenization technique the reader is referred to the book by Kalamkarov and Kolpakov (1997) for example).

The homogenization method is applied to composites with periodic microstructures. Thus

$$\Delta\mathbf{u}_\rho(\mathbf{X},\mathbf{Y}) = \Delta\mathbf{u}_\rho(\mathbf{X},\mathbf{Y} + n_p\mathbf{d}_p) \tag{33}$$

with $\rho = 0, 1, \ldots$, where n_p are arbitrary integer numbers and the constant vectors $\mathbf{d}_p$ determine the period of the structure.

Due to the change of coordinates from the global to the local systems the following relation must be employed in evaluating the derivative of a field quantity:

$$\frac{\partial}{\partial\mathbf{X}} \to \frac{\partial}{\partial\mathbf{X}} + \frac{1}{\delta}\frac{\partial}{\partial\mathbf{Y}} \tag{34}$$

The quantity $\Delta\mathbf{u}_0$ is the displacement increment in the homogenized region and hence it is not functions of $\mathbf{Y}$. Let

$$\Delta\mathbf{u}_0 = \Delta\mathbf{u}_0(\mathbf{X}) \equiv \Delta\bar{\mathbf{u}} \tag{35}$$

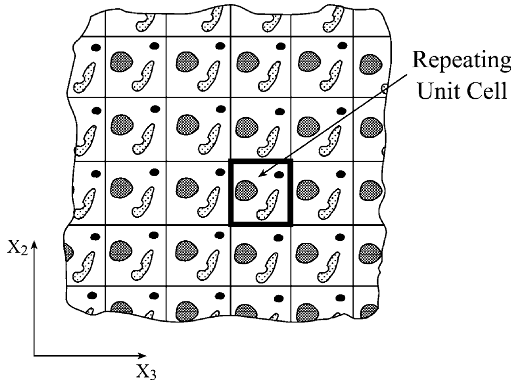

Fig. 1. A multiphase composite with a periodic microstructure in the $X_2 - X_3$ plane characterized by a repeating unit cell (highlighted).

and

$$\Delta \mathbf{u}_1 \equiv \Delta \tilde{\mathbf{u}}(\mathbf{X}, \mathbf{Y}) \tag{36}$$

where the latter is the fluctuating displacement increment, which is unknown periodic function with respect to $\mathbf{Y}$. This displacement increment arises due to the heterogeneity of the medium.

The increment of the deformation gradient is determined from the displacement expansion increment, Eq. (31), yielding the following expression

$$\Delta \mathbf{F} = \Delta \bar{\mathbf{F}}(\mathbf{X}) + \Delta \tilde{\mathbf{F}}(\mathbf{X}, \mathbf{Y}) + O(\delta) \tag{37}$$

where

$$\Delta \bar{\mathbf{F}}(\mathbf{X}) = \Delta \frac{\partial \bar{\mathbf{u}}}{\partial \mathbf{X}} \tag{38}$$

and

$$\Delta \tilde{\mathbf{F}}(\mathbf{X}, \mathbf{Y}) = \Delta \frac{\partial \tilde{\mathbf{u}}}{\partial \mathbf{Y}} \tag{39}$$

This shows that the increment of the deformation gradient can be represented as a sum of the deformation gradient increment $\Delta \bar{\mathbf{F}}(\mathbf{X})$ in the composite and the fluctuating deformation gradient increment $\Delta \tilde{\mathbf{F}}(\mathbf{X}, \mathbf{Y})$.

For a composite that is subjected to homogeneous deformation, one can use Eq. (37) to represent the displacement increment in the form

$$\Delta \mathbf{u}(\mathbf{X}, \mathbf{Y}) = \Delta \bar{\mathbf{F}} \mathbf{X} + \Delta \tilde{\mathbf{u}} + O(\delta^2) \tag{40}$$

which is employed in constructing an approximate displacement field of the cell problem discussed in the sequel.

For specified values of the average displacement gradient increment, the unknown fluctuating displacement increments are governed by the equilibrium equations subject to periodic boundary conditions imposed on the displacement and traction increments that are prescribed at the boundaries of the repeating unit cell. In addition, the continuity of displacement and traction increments at the internal interfaces are imposed between the phases within the repeating unit cell. The manner of solving the governing equations for the fluctuating displacement increments in the repeating unit cell is briefly described next. The details of the analysis for the more general three-dimensional case has been provided by Aboudi (2002).

4. Analysis of the repeating unit cell

The local analysis is performed on the repeating unit cell which is discretized into generic cells as shown in Fig. 2 for the repeating unit cell highlighted in Fig. 1. The repeating unit cell extends initially over the region $0 \leqslant Y_2 \leqslant H$, $0 \leqslant Y_3 \leqslant L$. The rectangular grid consists of N_q and N_r generic cells, where $q = 1, 2, \ldots, N_q$ and $r = 1, 2, \ldots, N_r$, identify the generic cell in the $Y_2 - Y_3$ plane. Each generic cell is further subdivided into four subcells designated by the pair (β, γ) where $\beta, \gamma = 1, 2$. The dimensions of the generic cell along the Y_2 and Y_3 axes are $h_1^{(q)}, h_2^{(q)}$ and $l_1^{(r)}, l_2^{(r)}$, respectively, such that

$$H = \sum_{q=1}^{N_q} (h_1^{(q)} + h_2^{(q)}), \quad L = \sum_{r=1}^{N_r} (l_1^{(r)} + l_2^{(r)})$$

Following the general displacement field representation for periodic media based on a two-scale expansion, the subcell displacement increment is approximated as follows (omitting the cell label (q, r)):

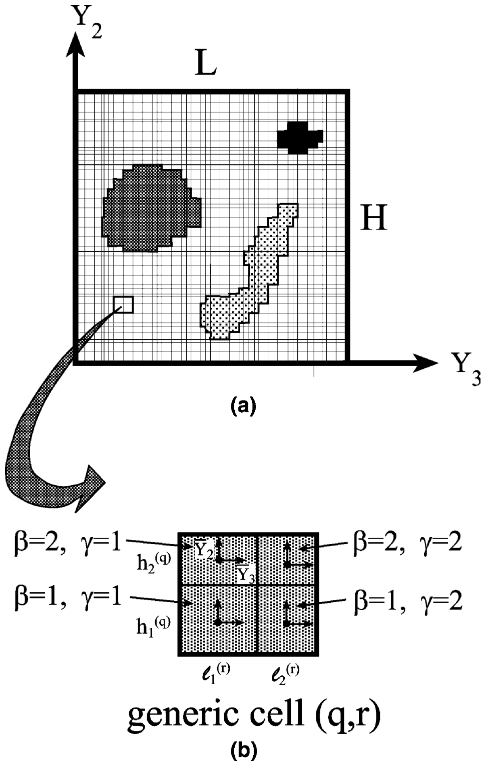

Fig. 2. (a) Discretization of the repeating unit cell employed in the present model, (b) generic cell within the repeating unit cell.

$$\begin{aligned}\Delta \mathbf{u}^{(\beta\gamma)} = {} & \Delta\bar{\mathbf{F}}\mathbf{X} + \Delta\mathbf{W}^{(\beta\gamma)}_{(00)} + \bar{Y}_2^{(\beta)}\Delta\mathbf{W}^{(\beta\gamma)}_{(10)} + \bar{Y}_3^{(\gamma)}\Delta\mathbf{W}^{(\beta\gamma)}_{(01)} + \frac{1}{2}\left(3\bar{Y}_2^{(\beta)2} - \frac{h_\beta^{(q)2}}{4}\right)\Delta\mathbf{W}^{(\beta\gamma)}_{(20)} \\ & + \frac{1}{2}\left(3\bar{Y}_3^{(\gamma)2} - \frac{l_\gamma^{(r)2}}{4}\right)\Delta\mathbf{W}^{(\beta\gamma)}_{(02)}\end{aligned} \tag{41}$$

where $\Delta\mathbf{W}^{(\beta\gamma)}_{(00)}$, which are the increments of the fluctuating volume-averaged displacements, and the higher-order terms $\Delta\mathbf{W}^{(\beta\gamma)}_{(mn)}$ must be determined from the governing equations, the periodic boundary conditions that the fluctuating mechanical field must fulfill, and the interfacial continuity conditions between subcells all of which are imposed in a surface-average sense. Application of the above equations and conditions produces a system of $60N_qN_r$ algebraic equations in the unknowns $\Delta\mathbf{W}^{(\beta\gamma)}_{(mn)}$ of the form

$$\mathbf{K}\Delta\mathbf{U} = \Delta\mathbf{f} \tag{42}$$

where the structural stiffness matrix **K** contains information on the geometry and current tangent tensors of the materials within the individual subcells $(\beta\gamma)$ within the cells comprising the multiphase periodic composite. The increment of the displacement vector $\Delta\mathbf{U}$ contains the unknown displacement increment coefficients in each subcell, i.e.,

$$\Delta\mathbf{U} = [\Delta\mathbf{U}^{(11)}_{11}, \ldots, \Delta\mathbf{U}^{(22)}_{N_qN_r}] \tag{43}$$

where in subcell $(\beta\gamma)$ of cell (q, r) these coefficients are

$$\Delta\mathbf{U}^{(\beta\gamma)}_{qr} = \left[\Delta\mathbf{W}_{(00)}, \Delta\mathbf{W}_{(10)}, \Delta\mathbf{W}_{(01)}, \Delta\mathbf{W}_{(20)}, \Delta\mathbf{W}_{(02)}\right]^{(\beta\gamma)}_{qr}$$

The force vector $\Delta\mathbf{f}$ contains information on the externally applied deformation gradient increment $\Delta\bar{\mathbf{F}}$, the thermal stress tensors $\mathbf{H}_{qr}^{(\beta\gamma)}$, the temperature deviation $\Delta\theta$ and the tensor of viscoelastic increments $\Delta\mathbf{V}_{qr}^{(\beta\gamma)}$ in the subcell $(\beta\gamma)$ of cell (q,r).

5. The homogenized constitutive equations

The homogenized constitutive equations of the multiphase composite are obtained as relations between the average first Piola–Kirchhoff stress tensor increment $\Delta\bar{\mathbf{T}}$, the applied deformation gradient increment $\Delta\bar{\mathbf{F}}$, the temperature increment $\Delta\theta$ and the global viscoelastic stress increment $\Delta\bar{\mathbf{V}}$ as follows.

Once the solution of Eq. (42) has been established, one can proceed and determine the concentration tensors associated with the defined repeating unit cell. These tensors express the local increment of the deformation gradient in the cell $\Delta\mathbf{F}^{(\beta\gamma)}$ in terms of the increment of the global applied external deformation gradient $\Delta\bar{\mathbf{F}}$, temperature increment $\Delta\theta$ and of the viscoelastic deformation gradient increment $\Delta\mathbf{F}_V^{(\beta\gamma)}$:

$$\Delta\mathbf{F}^{(\beta\gamma)} = \mathbf{A}^{(\beta\gamma)}\Delta\bar{\mathbf{F}} + \mathbf{a}^{(\beta\gamma)}\Delta\theta + \Delta\mathbf{F}_V^{(\beta\gamma)} \tag{44}$$

In this equation, $\mathbf{A}^{(\beta\gamma)}$ and $\mathbf{a}^{(\beta\gamma)}$ represent the mechanical and thermal concentration tensors, respectively, while $\Delta\mathbf{F}_V^{(\beta\gamma)}$ is the contribution of the increments of the viscoelastic effects to the local deformation gradient in the repeating unit cell. The instantaneous mechanical concentration tensor $\mathbf{A}^{(\beta\gamma)}$ is computed from solution of Eq. (42) by imposing $\Delta\theta = 0$ and in the absence of viscoelastic effects, while the current thermal concentration vector $\mathbf{a}^{(\beta\gamma)}$ is computed from the solution of Eq. (42) when $\Delta\bar{\mathbf{F}} = \mathbf{0}$ and in the absence of viscoelastic effects. Finally, the current $\Delta\mathbf{F}_V^{(\beta\gamma)}$ is computed from the solution of Eq. (42) when $\Delta\bar{\mathbf{F}} = \mathbf{0}$ and $\Delta\theta = 0$ are imposed.

The stress increment in the subcell is given according to constitutive equation (27) by

$$\Delta\mathbf{T}^{(\beta\gamma)} = \mathbf{R}^{(\beta\gamma)}\Delta\mathbf{F}^{(\beta\gamma)} - \mathbf{H}^{(\beta\gamma)}\Delta\theta - \Delta\mathbf{V}^{(\beta\gamma)} \tag{45}$$

Applying the definition of the average increment of the first Piola–Kirchhoff stress tensor over the repeating unit cell in terms of the subcell stress increment yields

$$\Delta\bar{\mathbf{T}} = \frac{1}{HL}\sum_{q=1}^{N_q}\sum_{r=1}^{N_r}\sum_{\beta,\gamma=1}^{2} h_\beta^{(q)} l_\gamma^{(r)} [\Delta\mathbf{T}^{(\beta\gamma)}]^{(q,r)} \tag{46}$$

Consequently, the use of Eqs. (44) and (45) in (46) yields the micromechanically based global constitutive relations

$$\Delta\bar{\mathbf{T}} = \mathbf{R}^*\Delta\bar{\mathbf{F}} - \mathbf{H}^*\Delta\theta - \Delta\bar{\mathbf{V}} \tag{47}$$

where $\mathbf{R}^*$ and $\mathbf{H}^*$ are the instantaneous effective stiffness and effective thermal stress tangent tensors of the composite which are given by

$$\mathbf{R}^* = \frac{1}{HL}\sum_{q=1}^{N_q}\sum_{r=1}^{N_r}\sum_{\beta,\gamma=1}^{2} h_\beta^{(q)} l_\gamma^{(r)} \left[\mathbf{R}^{(\beta\gamma)}\mathbf{A}^{(\beta\gamma)}\right]^{(q,r)} \tag{48}$$

$$\mathbf{H}^* = -\frac{1}{HL}\sum_{q=1}^{N_q}\sum_{r=1}^{N_r}\sum_{\beta,\gamma=1}^{2} h_\beta^{(q)} l_\gamma^{(r)} \left[\mathbf{R}^{(\beta\gamma)}\mathbf{a}^{(\beta\gamma)} - \mathbf{H}^{(\beta\gamma)}\right]^{(q,r)} \tag{49}$$

and the increment of the global viscoelastic stress tensor is determined from

$$\Delta\bar{\mathbf{V}} = -\frac{1}{HL}\sum_{q=1}^{N_q}\sum_{r=1}^{N_r}\sum_{\beta,\gamma=1}^{2} h_\beta^{(q)} l_\gamma^{(r)} \left[\mathbf{R}^{(\beta\gamma)}\Delta\mathbf{F}_V^{(\beta\gamma)} - \Delta\mathbf{V}^{(\beta\gamma)}\right]^{(q,r)} \tag{50}$$

Constitutive relations (47) are the micromechanically established equations that govern the global behavior of thermoviscoelastic multiphase composite undergoing large deformations in which any one of the phases can be assumed, in general, to be modeled by the incremental thermoviscoelastic equations (27). Elastic, viscoelastic and thermoelastic behavior multiphase composites at finite strains can be obtained as special cases of Eq. (47) by an appropriate specialization the constitutive law of the phases.

It should be noted that the instantaneous effective stiffness tensor $\mathbf{R}^*$ describes the current anisotropic behavior of the periodic multiphase material. The specific type of the resulting anisotropy depends on the microstructural architecture of the repeating unit cell.

6. Applications

Let us illustrate the implementation of the established constitutive relations (47) by investigating the response of a composite material that consists of a rubber-like matrix reinforced by nylon fibers. The material properties of a thermoelastic vulcanized rubber at a reference temperature $\theta_0 = 293$ K are given in Table 1. These properties characterize the long-term behavior of the material. In order to include its viscoelastic response, the following parameters have been assumed:

$$\omega_\alpha = \frac{1}{2}, \quad \tau_\alpha = 10^{(\alpha-2)}, \quad \alpha = 1,\ldots,N_\alpha = 6$$

The nylon fibers have been assumed to behave as a linearly elastic material with Young's modulus, Poisson's ratio and coefficient of thermal expansion, given, respectively, by (Perry, 1963): 2 GPa, 0.4, 90×10^{-6}/K. The orientation of the fibers is selected to be initially in the X_1-direction and their volume fraction is $v_f = 0.25$.

Results are given for a rubber-like matrix composite that is subjected to:

(1) thermal loading with prescribed transverse stretch,
(2) thermal loading with prescribed transverse stress,
(3) free thermal loading,

Table 1
Material properties of vulcanized rubber at $\theta_0 = 293$ K (Chadwick, 1974; Morman, 1995)

Property	Value
κ	1950 MPa
G	0.98 MPa
m	9
n	2.5
γ	$\frac{1}{6}$
α_0	657×10^{-6} /K
c_{10}	0.2357 MPa
c_{01}	0.2426 MPa
c_{11}	−0.567 kPa
c_{20}	7.41 kPa
c_{30}	0.0464 kPa

(4) transverse uniaxial mechanical loading at room and elevated temperature,
(5) creep caused by the application of a constant transverse uniaxial stress loading at room and elevated temperature,
(6) relaxation caused by the application of a constant transverse stretch at room and elevated temperature.

Cases (1) and (2) will exhibit the anomalous behavior of the rubber, while case (3) will exhibit the response in the axial direction (i.e., in the fiber direction) and transverse direction (i.e., perpendicular to the fiber), and case (4) will exhibit the hysteresis due to the viscoelastic effects and the response rate-dependence. In all cases the effect of the reinforcement is exhibited by comparison with the corresponding unreinforced ($v_f = 0$) thermoviscoelastic rubber's response.

Fig. 3 exhibits the response of the unreinforced rubber-like matrix and the unidirectional composite caused by the application of a thermal loading in which the temperature is incrementally increased from $\theta_0 = 293$ K by 150 K. In addition, constant prescribed stretches $\overline{F}_{22} = 1.01$ and 1.3 are applied in the transverse direction. Both the thermoviscoelastic and long-term thermoelastic responses are shown. For the low stretch value $\overline{F}_{22} = 1.01$, the ordinary behavior both in the unreinforced matrix and composite is obtained in which the stress decreases with temperature, namely the stress gradient is negative. For the high value of the stretch $\overline{F}_{22} = 1.3$, this gradient is positive in the case of the unreinforced matrix. But in the composite material with the thermoviscoelastic matrix this trend is reversed yielding again the ordinary behavior. Thus the combined effect of reinforcement and viscoelastic mechanism has a significant influence on the material's thermal response.

The resulting responses caused by aforementioned thermal loading applied on the unreinforced rubber-like matrix and the unidirectional composite, in conjunction with prescribed transverse stress $\overline{T}_{22}$ of 0.1 and 1 MPa, are shown in Fig. 4. For the lower value of the applied stress, the ordinary behavior where positive stretch gradients with respect to the temperature are obtained both in the thermoviscoelastic and the long-term thermoelastic cases. For the higher value of the applied stress, $\overline{T}_{22}$ of 1 MPa, this gradient is reversed in the elastic case thus exhibiting the anomalous behavior of the unreinforced rubber. As is observed from this figure, the latter behavior is lost, however, in the presence of viscoelastic effects where the response is similar to that exhibited by ordinary materials. In conclusion, Figs. 3 and 4 show that for the chosen material and the associated viscoelastic parameters, the viscoelastic mechanism has a peculiar effect on the anomalous behavior of the homogeneous rubber-like materials and unidirectional rubber-like matrix

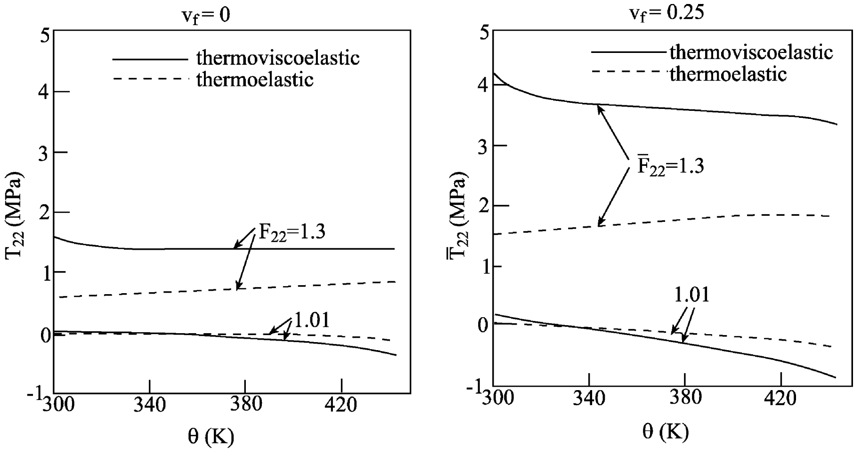

Fig. 3. Stress–temperature response of homogeneous matrix ($v_f = 0$) and unidirectional nylon/rubber composite that are subjected to two values of prescribed stretches in the transverse 2-direction.

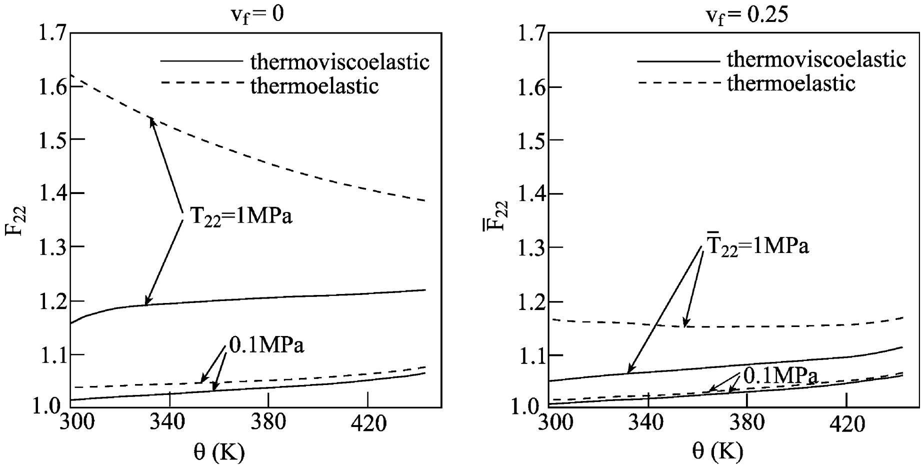

Fig. 4. Stretch–temperature response of homogeneous matrix ($v_f = 0$) and unidirectional nylon/rubber composite that are subjected to two values of prescribed normal stresses in the transverse 2-direction.

composites. This peculiar thermomechanical behavior of rubber, which is referred to as the Gough–Joule effect, depends on the fiber volume fraction, loading direction and viscoelastic mechanism of the constituents.

Let us consider next the case of a free thermal expansion. Here the composite is subjected to an incremental temperature rise of 150 K from θ_0 while keeping it stress-free, namely $\overline{T}_{ij} = 0$. The resulting axial $\overline{F}_{11}$ and transverse $\overline{F}_{22} = \overline{F}_{33}$ stretches are shown in Fig. 5 for the homogeneous matrix and unidirectional composite. The effect of reinforcement on the induced stretches is clearly exhibited in this figure. In linearly elastic materials and composites the stretch–temperature dependence is linear. Consequently, the effect of the existing non-linear behavior of the matrix on the composite responses can be clearly observed from the figure. The linear axial stretch–temperature response exhibited by the composite results from the dominant effect of the linearly elastic nylon fibers. This follows from the high contrast between the nylon

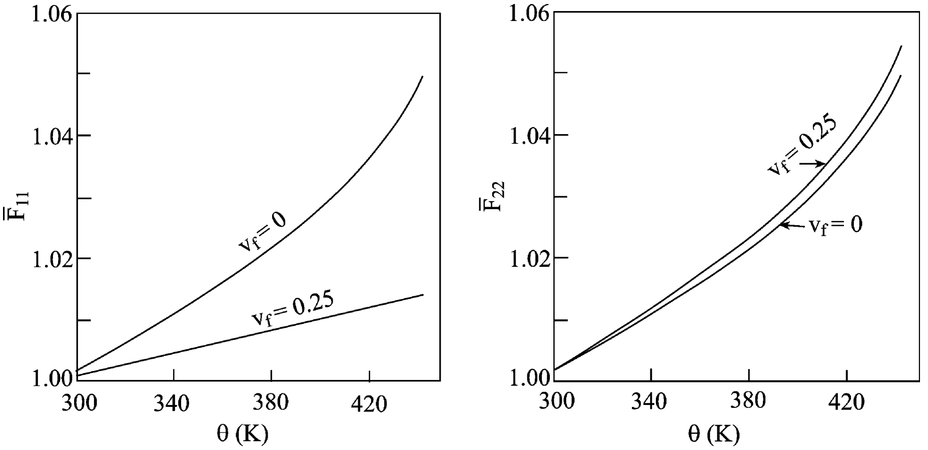

Fig. 5. Axial and transverse stretches against temperature due to the free thermal expansion of the thermoviscoelastic unreinforced matrix ($v_f = 0$) and unidirectional composite.

and rubber Young's moduli which renders the effect of the matrix on the response in the axial direction (parallel to the fiber) to be negligible. In the transverse direction (perpendicular to the fiber) the effect of matrix is seen to be dominant. It should be mentioned that as long as a free thermal expansion is considered, the effect of viscoelastic mechanism has been found to be relatively weak so that the curves shown in Fig. 5 almost coincide with the corresponding curves computed from the long-term material's thermoelastic constitutive law. As a result, the response does not exhibit rate-dependent effects.

The response of the composite at room and elevated temperatures caused by the application of a uniaxial transverse stress loading in the 2-direction at two rates is shown in Figs. 6 and 7. Both figures show the resulting transverse stress $\overline{T}_{22}$ due to a complete cycle of loading–unloading–reloading. The figures exhibit the effect of reinforcement (by comparison with the response of the homogeneous matrix), the effect of loading at two values of strain rates: $\dot{\bar{F}}_{22} = 0.001$ and 1 s^{-1}, and the effect of loading at two temperatures. In

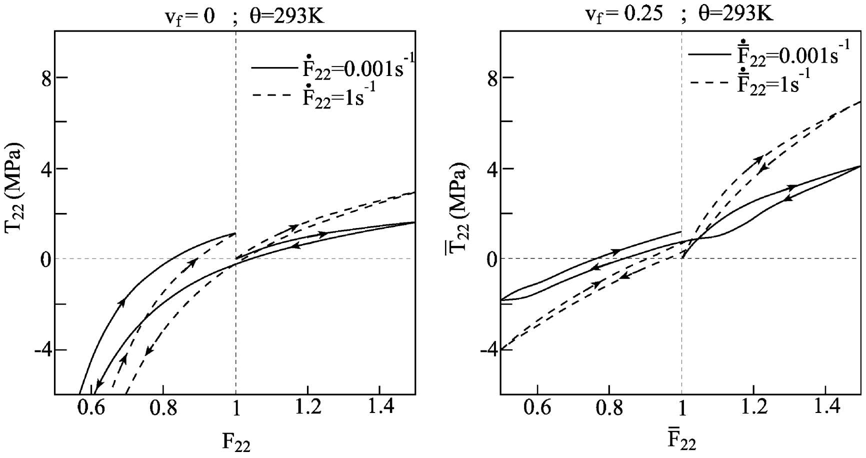

Fig. 6. Stress response at room temperature of the thermoviscoelastic homogeneous matrix ($v_f = 0$) and unidirectional composite to a complete cycle of transverse uniaxial stress loading applied at two rates.

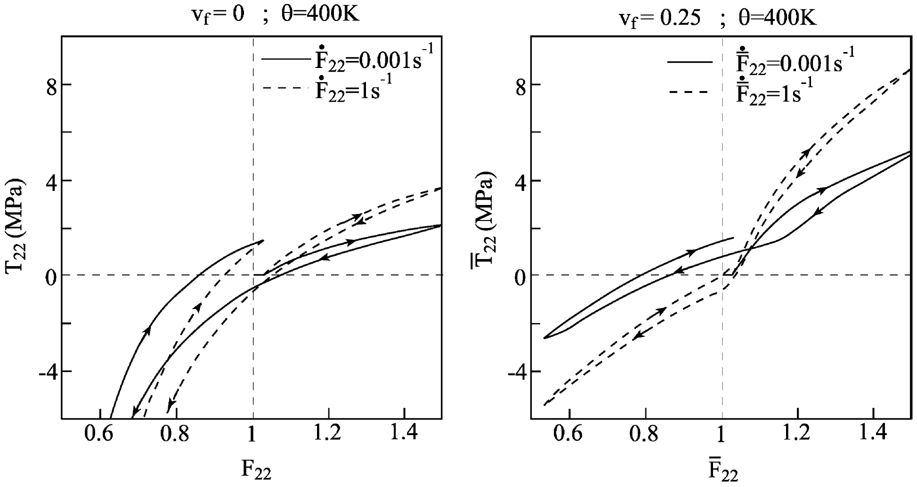

Fig. 7. Stress response at elevated temperature ($\theta = 400$ K) of the thermoviscoelastic homogeneous matrix ($v_f = 0$) and unidirectional composite to a complete cycle of transverse uniaxial stress loading applied at two rates.

addition, the figures clearly exhibit the hysteresis loops caused by the viscoelastic mechanism. as well as the effect of temperature on these loops. It should be noted that due to the process of free temperature loading, which is required to bring the composite to the elevated temperature $\theta = 400$ K prior to the application of mechanical loading, there is a slight shift in the plot of the response in Fig. 7 before the starting point of the cycle, as a result of which the entire graph is shifted.

It is also possible to detect the effect of reinforcement and elevated temperature on the composite's response to the full cycle of uniaxial transverse stress loading in the 2-direction by plotting $\overline{F}_{33}$ against $\overline{F}_{22}$. This graph is shown in Fig. 8 where these effects are very well displayed. The aforementioned shift due to the thermal loading that is required to raise the composite's temperature from θ_0 to $\theta = 400$ K is clearly observed.

Figs. 6–8 exhibit the response to one cycle of transverse loading of the matrix and composite. Let us track the response to a several cycles by plotting it against time. In Fig. 9, the behavior of the rubber-like matrix and the composite is shown at room and elevated temperature $\theta = 400$ K due to five cycles of uniaxial transverse stress loading applied at a rate of $\dot{\overline{F}}_{22} = 0.001$ s^{-1}. The graphs show the effect of reinforcement, viscous mechanism and temperature values. The increase with temperature of the bulk and shear moduli of the rubber matrix, results into the observed increase of the response amplitudes at elevated temperature. Here too, the shift observed in the response at elevated temperature can be well observed. A careful examination of the repeated patterns in Fig. 9 displays the attenuation effect caused by the viscoelastic mechanism.

The effect of viscoelastic mechanisms at room and elevated temperature can be well exhibited by displaying the creep and relaxation behavior of the unreinforced rubber-like matrix and nylon/matrix composite. To this end, Figs. 10 and 11 show the resulting response to a uniaxial transverse stress loading of $\overline{T}_{22} = 2$ MPa and a transverse stretch of $\overline{F}_{22} = 1.5$, respectively, at room and elevated temperature ($\theta = 400$ K). Both figures show the variation of the response with time toward the final long-time values which are

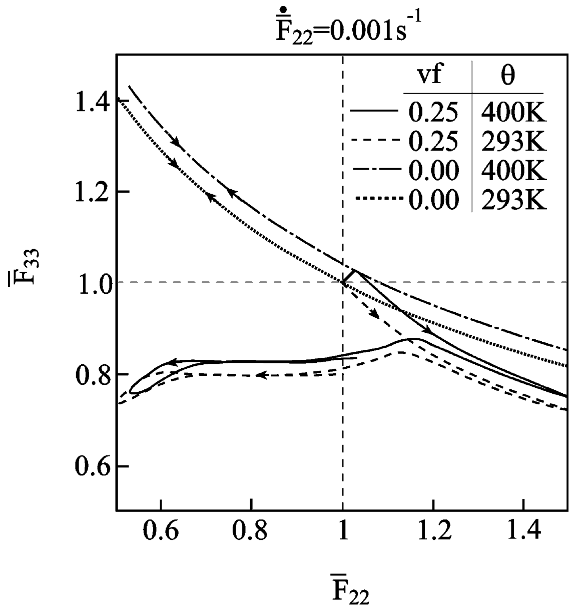

Fig. 8. The plot of $\overline{F}_{33}$ against $\overline{F}_{22}$ at room and elevated temperatures ($\theta = 400$ K) of the thermoviscoelastic homogeneous ($v_f = 0$) and unidirectional composite in the case of the application of a complete cycle of transverse uniaxial stress loading applied at a rate of 0.001 s^{-1}.

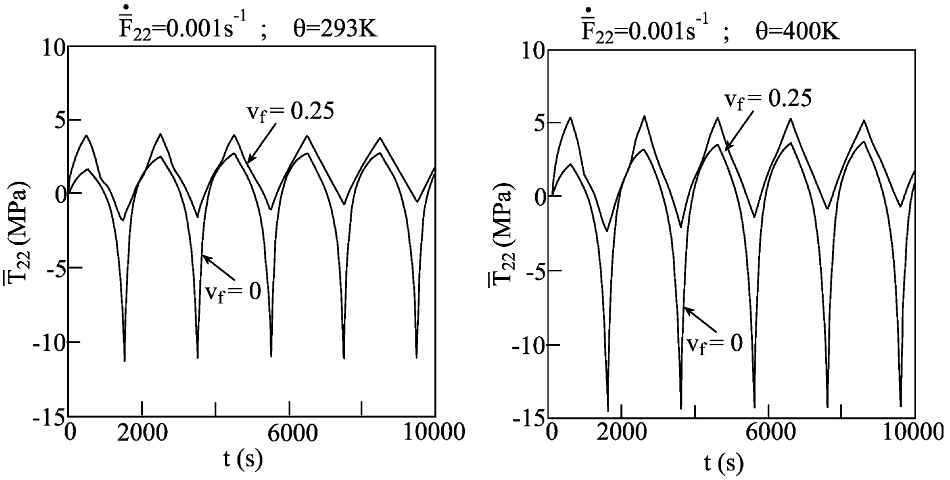

Fig. 9. Stress response at room and elevated temperatures ($\theta = 400$ K) against time of the thermoviscoelastic homogeneous matrix ($v_f = 0$) and unidirectional composite to five cycles of transverse uniaxial stress loading applied at a rate of 0.001 s^{-1}.

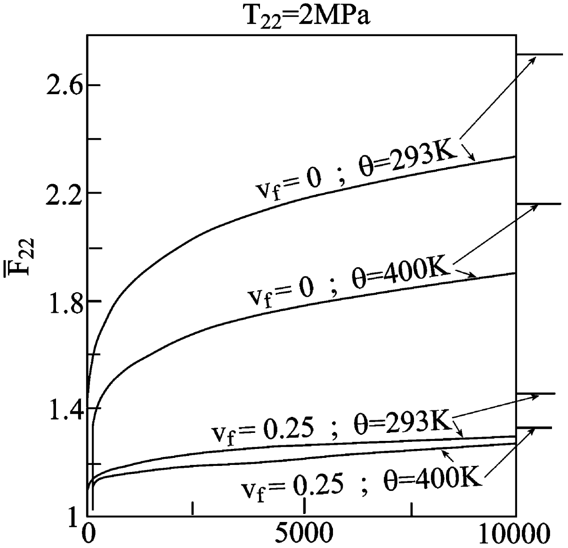

Fig. 10. Creep behavior of the thermoviscoelastic homogeneous matrix ($v_f = 0$) and unidirectional composite caused by the application of a constant uniaxial transverse stress loading of $\bar{T}_{22} = 2$ MPa at room and elevated temperature ($\theta = 400$ K). The asymptotic values are represented by horizontal lines.

depicted by horizontal lines. Both figures show that these asymptotic values are much faster approached by in the case of the unreinforced matrix than that of the composite (due to the reinforcing effect of the nylon fibers). In both figures, the shift due to the initial thermal loading which is observed in the creep and relaxation curves at elevated temperature can be clearly noticed.

In concluding this section it should be mentioned that in the above examples that illustrate the composite thermoviscoelastic response, the relaxation times have been taken as temperature-independent. In real

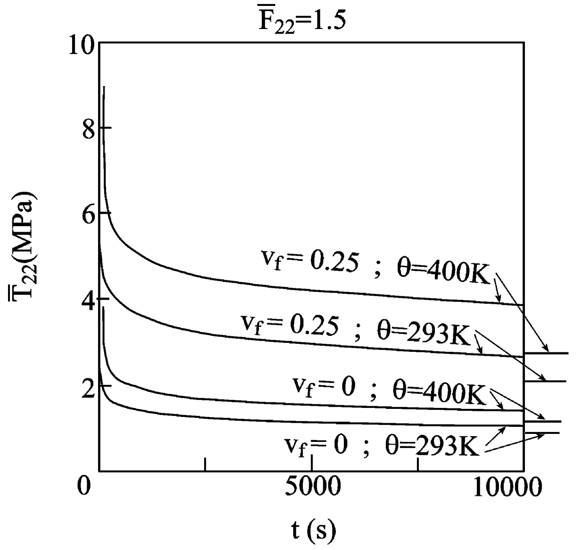

Fig. 11. Relaxation behavior of the thermoviscoelastic homogeneous matrix ($v_f = 0$) and unidirectional composite caused by the application of a constant uniaxial transverse stretch loading of $\overline{F}_{22} = 1.5$ at room and elevated temperature ($\theta = 400$ K). The asymptotic values are represented by horizontal lines.

situation the relaxation times decrease as the temperature increases. This implies that faster viscoelastic effects should be anticipated at elevated temperatures than those shown in these examples.

7. Conclusions

In the present investigation, a micromechanical theory based on the homogenization technique has been derived in order to establish the global constitutive equations of composites with thermoviscoelastic rubber-like materials undergoing finite strains. The absence of viscoelastic effects results in composites with thermoelastic rubber-like phases that are necessarily described by entropic elasticity.

One of the main advantages of the use of micromechanics is that the establishment of composite response and its anisotropic behavior are obtained as byproducts of the analysis. Thus, one does not need to consider anisotropic free-energy functions in order to model, directly, the macro behavior of the thermoviscoelastic multiphase material which is obviously a formidable task. In addition, residual stresses in the composite can be easily obtained by applying, prior to the application of mechanical loadings, a thermal loading that models the thermal processing.

In the present derivation, a one-way coupling exists between the thermal and mechanical fields namely, the temperature influences the stresses as in thermal stress problems. Generalization to full thermomechanical coupling (two-way coupling) which incorporates the influence of deformation on the thermal field is possible. The derived constitutive law can be applied to study the response of composite structures, such as plates and shells, with rubber-like thermoviscoelastic materials.

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